CORRECTION TO BLACK-SCHOLES FORMULA DUE TO FRACTIONAL STOCHASTIC VOLATILITY

JOSSELIN GARNIER* AND KNUT SØLNA†

Abstract. Empirical studies show that the volatility may exhibit correlations that decay as a fractional power of the time offset. The paper presents a rigorous analysis for the case when the stationary stochastic volatility model is constructed in terms of a fractional Ornstein Uhlenbeck process to have such correlations. It is shown how the associated implied volatility has a term structure that is a function of maturity to a fractional power.

Key words. Stochastic volatility, implied volatility, fractional Brownian motion, long-range dependence.

AMS subject classifications. 91G80, 60H10, 60G22, 60K37.

1. Introduction.

Our aim in this paper is to provide a framework for analysis of stochastic volatility problems in the context when the volatility process possesses long-range correlations. Replacing the constant volatility of the Black-Scholes model with a random process gives price modifications in financial contracts. It is important to understand the qualitative behavior of such price modifications for a (class of) stochastic volatility models since this can be used for calibration purposes. Typically the price modifications are parameterized by the implied volatility relative to the Black-Scholes model [25, 41]. For illustration we consider here European option pricing and then the implied volatility depends on the moneyness, the ratio between the strike price and the current price, moreover, the time to maturity. The term and moneyness structure of the implied volatility can be calibrated with respect to liquid contracts and then used for pricing of related but less liquid contracts. Much of the work on stochastic volatility models have focussed on situations when the volatility process is a Markov process, commonly some sort of a jump diffusion process. However, a number of empirical studies suggest that the volatility process possesses long- and short-range dependence, that is the correlation function of the volatility process has decay that is a fractional power of the time offset. This is the class of volatility models we consider here. We find that such correlations indeed reflect themself in an implied volatility fractional term structure. An important aspect of the modeling is also the presence of correlation between the volatility shocks and the shocks (driving Brownian motion) of the underlying, this "leverage effect" influences the implied volatility in an important way and we shall include it below. The leverage effect is well motivated from the modeling viewpoint and important to incorporate to fit observed implied volatilities, albeit a challenging quantity to estimate [2]. Evidence of leverage and persistence or long-range dependencies have been found by considering high-frequency data and incorporated in discrete time series models [8, 20, 42].

Here we model in terms of a continuous time stochastic volatility model that is a smooth function of a Gaussian process. We use a martingale method approach which exploits the fact that the discounted price process is a (local) martingale. We model the fractional stochastic volatility (fSV) as a smooth function of a fractional Ornstein-Uhlenbeck (fOU) process. We moreover assume that the fSV model has

*Laboratoire de Probabilités et Modèles Aléatoires & Laboratoire Jacques-Louis Lions, Université Paris Diderot, 75205 Paris Cedex 13, France garnier@math.univ-paris-diderot.fr

†Department of Mathematics, University of California, Irvine CA 92697 ksolna@math.uci.edu

relatively small fluctuations, of magnitude $\delta \ll 1$ and we derive the associated leading order expression for implied volatility with respect to this parameter via an asymptotic analysis. This gives a parsimonious parameterization of the implied volatility which may be exploited for robust calibration. The fOU process is a classic model for a stationary process with a fractional correlation structure. This process can be expressed in terms of an integral of a fractional Brownian motion (fBm) process. The distribution of a fBm process is characterized in terms of the Hurst exponent $H \in (0,1)$. The fBm process is locally Hölder continuous of exponent α for all $\alpha < H$ and this property is inherited by the fOU process. The fBm process, W_t^H, is also self-similar in that

$$\left\{W_{\alpha t}^H, t \in \mathbb{R}\right\} \stackrel{dist.}{=} \left\{\alpha^H W_t^H, t \in \mathbb{R}\right\} \text{ for all } \alpha > 0. \tag{1.1}$$

The self-similarity property is inherited approximately by the fOU process on scales smaller than the mean reversion time of the fOU process that we will denote by $1/a$ below. In this sense we may refer to the fOU process as a multiscale process on relatively short scales.

The case with $H \in (0, 1/2)$ gives a fOU process that is a so-called "short-range" dependent process that is rough on short scales and whose correlations for small time offsets decay faster than the linear decay associated with a Markov process. In fact the decay is as the offset to the fractional power $2H$. In this regime consecutive increments of the fBm process are negatively correlated giving a rough process also referred to as an anti-persistent process. The enhanced negative correlation with smaller Hurst exponent gives a relatively rougher process.

The case with $H \in (1/2, 1)$ gives a fOU process that is a so-called "long-range" dependent process whose correlations for large time offsets decays as the offset to the fractional power $2(H-1)$. It follows that the correlation function of the fOU process is not integrable. This regime corresponds to a persistent process where consecutive increments of the fBm are positively correlated. The relatively stronger positive correlation for the consecutive increments of the associated fBm process with increasing H values gives a relatively smoother process whose correlations decay relatively slowly. For more details regarding the fBm and fOU processes we refer repectively to [7, 17, 18, 36] and [10, 34].

A large number of recent papers have considered modeling of volatility in terms of processes with short- and long-range dependence. In [13] the authors consider a long memory extension of the Heston [33] option pricing model, a fractionally integrated square root process, a generalization of the early work in [14]. They make use of the analytical tractability of this model, in fact a fractionally integrated version of a Markovian affine diffusion, with affine diffusions considered in [19]. The emphasis is on the long-range dependent case ($H > 1/2$) and long time to maturity. The authors focus on the conditional expectation of the integrated square volatility and show the fractional decay of this, moreover, they discuss estimation schemes for model parameters based on discrete observations. In the Markovian case the mean integrated square volatility would exponentially fast approach its mean value and flatten the implied volatility term structure. They remark that long-range dependence provides an explanation for observations of non-flat term structure in the regime of large maturities since the long-range dependence may make the implied volatility smile strongly maturity-dependent in this regime, while also producing consistent smiles for short maturities. The model presented in [13] was recently revisited in [30] where short and long maturity asymptotics are analyzed using large deviations principles.

The concept of RFSV, Rough Fractional Stochastic Volatility, is put forward in [5, 29]. Here a model with log-volatility modeled by a fBm is motivated by analysis of market data, which they state provide strong support for a value for the Hurst exponent H around 0.1. As explained above small values for H correspond to very rough processes. It is remarked that such a process can be motivated by modeling of order flow using Hawkes processes. The authors discuss issues related to change from physical to pricing measure and use simulated prices to fit well the implied volatility surface in the case of SPX with few parameters. They argue that the fractional model generates strong skews or "smiles" in the implied volatility even for very short time to maturity so that this modeling provides an alternative to using jumps to model such an effect. The form of the implied volatility surface and the structure of the returns have been used to argue that the asset price should be a jump process [1, 9]. Indeed models with jumps may be used as an alternative approach to capture smile dynamics to the fractional approach considered here and recent contributions consider models driven by Lévy processes both for volatility models [21, 39] and directly for price models [4].

A variant of the model in [29] is considered in [37] where the log-volatility is modeled as a fractional noise, with fractional noise being the increment process of a fBm for a certain increment length. The authors discuss the well-posedness of this model from the financial perspective and in doing so make use of a truncated version of the integral representation of the fBm. In [38] this model is supported by data analysis and motivated by an agent-based interpretation.

In [11, 12] the authors consider the situation when the volatility is modeled as a function of a fOU process whose shocks are independent from those of the underlying. Their focus is on a tree-based method for computing prices, estimation schemes for model parameters, and a particle filtering technique for the unobserved volatility given discrete observations. They consider some real data examples and find estimated values for the Hurst exponent which is larger than $1/2$, in particular in a period after a market crash. In [31] small maturity asymptotic results are presented for this model.

Among the many papers considering short maturity asymptotics, in the early paper [3] Alós et al. use Malliavin calculus to get expressions for the implied volatility in the regime of small maturity. They find that the implied volatility diverges in the short-range dependent case and flattens in the long-range dependent case in the limit of small maturity. These results are consistent with what we present below. The modeling in [3] differs from the modeling below in that the authors consider volatility fluctuations at the order one level while below the fluctuations are relatively small, however, we consider any time to maturity.

Fukasawa [27] discusses the case with small volatility fluctuations and short- and long-range dependence impact on the implied volatility as an application of the general theory he sets forth. He uses a non-stationary "planar" fBm as the volatility factor so that the leading implied volatility surface is identified conditioned on the present value of the implied volatility factor only, while below with a stationary model the surface depends on the path of the volatility factor until the present, reflecting the non-Markovian nature of fBm. In [28] Fukasawa discusses the case of short-range dependent processes and short time to maturity and a framework for expansion of the implied volatility surface. He uses a representation of fBm due to Muralev [40]. He also considers local stochastic volatility models and find that these are not consistent with power laws in this regime.

As a further generalization relative to a fractional Brownian motion based model the case of multi fractional Brownian motion based models is considered in [16]. This allows for a non-stationary local regularity or a time dependent Hurst exponent and then the implied volatility depends on weighted averages of the local Hurst exponent.

In [23] Forde and Zhang use large deviation principles to compute the short maturity asymptotic form of the implied volatility. They consider the correlated case with leverage and obtain results that are consistent with those in [3]. They consider a stochastic volatility model based on fBm and also more general ones where the volatility process is driven by fBms and which is analyzed using rough path theory. They also consider large time asymptotics for some fractional processes.

Indeed, a number of recent papers have considered small maturity asymptotics for implied volatility in the context of mixing, short- or long-range processes. Many of these use large deviation principles or heat kernel expansions [6, 23, 32], while another approach is to consider the regime around the money [3, 28, 39]. Recent works deal also with the regime of large strikes and derive bounds on the implied volatility [35]. Here we take another approach by considering a perturbation situation so that the implied volatility can be expanded around an effective volatility [25], also for large times to maturity. We model the volatility as a stationary process, a continuous time stationary short- or long-range dependent stochastic volatility process, with the view toward constructing a time consistent scheme. We use an approach based on the martingale method which is adapted to the fact that the volatility process is not a Markov process. We explicitly take into account the effects of correlation in between volatility shocks and shocks in the underlying, the leverage effect, and its form in short- and long-range dependent cases. We obtain expressions for the implied volatility for all times to maturity and also for log-moneyness of order one. Explicitly, we model the volatility as

$$\sigma_t = \bar{\sigma} + F(\delta Z_t^H), \tag{1.2}$$

for Z_t^H the fOU process that we discuss in more detail in Section 2.2. The function F is assumed to be one-to-one, smooth, bounded from below by a constant larger than $-\bar{\sigma}$, with bounded derivatives, and such that $F(0) = 0$ and $F'(0) = 1$. It follows that the volatility process inherits (qualitatively) the correlation properties of the fBm process. Note that throughout we will be working with non-dimensionalized quantities.

The main result is then the associated form for the implied volatility, see Equations (5.1), (5.3) and (5.4) below, we summarize the result next. The implied volatility is here the volatility value that needs to be used in the constant volatility Black-Scholes European option pricing formula in order to replicate the asymptotic fSV option price, it is, up to terms of order δ^2:

$$I_t = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T \sigma_s^2 ds|\mathcal{F}_t\Big]^{\frac{1}{2}} + \mathcal{A}(T-t)\Big[1 + \frac{\log(K/X_t)}{(T-t)/\bar{\tau}}\Big], \tag{1.3}$$

for

$$\mathcal{A}(\tau) = \frac{\delta\rho\bar{\sigma}\tau^{H+\frac{1}{2}}}{2\Gamma(H+\frac{5}{2})}\Big\{1 - \int_0^{a\tau} e^{-v}\big(1-\frac{v}{a\tau}\big)^{H+\frac{3}{2}} dv\Big\}, \tag{1.4}$$

with $1/a$ is the mean reversion time of the fOU process and $\bar{\tau} = 2/\bar{\sigma}^2$ a characteristic diffusion time for the underlying. Furthermore, X_t is the underlying price process with evolution as in (3.1) and $\mathcal{F}_t$ its associated filtration. Moreover, ρ is the correlation in between the Brownian motions driving respectively the volatility process and the underlying price process, K is the strike price so that K/X is the moneyness, and finally $\tau = T - t$ is time to maturity. The first term in the implied volatility is the expected effective volatility over the remaining time period of the option conditioned on the knowledge at time t, note that this term is random. The second term is a leverage term which is present in the case that the underlying and the volatility have correlated evolutions so that ρ is non-zero. Note that ρ is commonly assumed to be negative. The log-moneyness term becomes relatively more important as the time to maturity becomes small relative to the characteristic diffusion time.

In the short and long time to maturity regimes we then have for the leverage term:

$$\mathcal{A}(\tau)\Big[1 + \frac{\log(K/X_t)}{\tau/\bar{\tau}}\Big] = \left\{ \begin{array}{ll} a_{\rm s}\Big[(\tau/\bar{\tau})^{\frac{1}{2}+H} + (\tau/\bar{\tau})^{-\frac{1}{2}+H}\log(K/X_t)\Big] & \text{for } a\tau \ll 1, \\ a_{\rm l}\Big[(\tau/\bar{\tau})^{-\frac{1}{2}+H} + (\tau/\bar{\tau})^{-\frac{3}{2}+H}\log(K/X_t)\Big] & \text{for } a\tau \gg 1\,, \end{array} \right. \tag{1.5}$$

for

$$a_{\rm s} = \frac{\delta\rho\bar{\tau}^H}{\sqrt{2}\Gamma(H+\frac{5}{2})}, \qquad a_{\rm l} = \frac{\delta\rho\bar{\tau}^{H-1}}{\sqrt{2}a\Gamma(H+\frac{3}{2})}. \tag{1.6}$$

We moreover have for the predicted effective volatility term:

$$\sigma_{t,T} \equiv \mathbb{E}\Big[\frac{1}{T-t}\int_t^T \sigma_s^2 ds | \mathcal{F}_t\Big]^{\frac{1}{2}} = \left\{ \begin{array}{ll} \sigma_t & \text{for } a\tau \ll 1, \\ \bar{\sigma} & \text{for } a\tau \gg 1. \end{array} \right. \tag{1.7}$$

It is important to note that we only assume $\tau = T - t > 0$ so that in fact the implied volatility for small times to maturity may be very large for short-range dependent processes. This reflects the fact that for short-range dependent processes the volatility path is rough and may have a significant impact beyond the current predicted effective volatility level. However, when used in the standard Black-Scholes pricing formula the implied volatility indeed gives a pricing correction that is $O(\delta)$ for any $\tau > 0$. We also note that in the long maturity regime the implied volatility level may diverge for long-range dependent processes reflecting the fact that long-range dependence gives strong temporal coherence and therefore relatively large corrections to the predicted current effective volatility.

Note next that the calibration of the leverage component of the implied volatility in the general case in (1.3) involves estimation of the group market parameters:

$$\bar{\sigma}, \quad H, \quad (\delta\rho), \quad a, \tag{1.8}$$

from observed implied volatility data. In order to fully identify the model at the current time t we need moreover to estimate the current predicted effective volatility over a time to maturity horizon, that is, $\sigma_{t,t+\tau}$ for $0 \le \tau \le T_{\max} - t$.

It is important to note that in our framework the market parameters are from the theoretical point of view independent of the current time t. Thus, in order to calibrate the model with data over a current time epoch $t_1 \le t \le t_2$ one may use all the implied volatility recording in a joint fitting procedure.

We remark that our results would be modified under the presence of general interest rates and market price of risk factors that we do not consider here. We also remark that identifying a "smile" shape, that is a more general function in log-moneyness, would require a higher-order approximation of implied volatility [26]. Finally, observe that the case $H = 1/2$ corresponds neither to a short-range dependent process nor a long-range dependent process, but the standard case of an Ornstein-Uhlenbeck process and a stochastic volatility that is a Markovian process with correlations decaying exponentially fast [25].

The framework we have presented is general and can be used for processes for which we can identify the key quantities of interest below. We discuss one important special case corresponding to a slow fOU process. In this case we model the volatility in terms of the "slow" fOU process $Z^{\delta,H}$:

$$Z_t^{\delta,H} = \delta^H \int_{-\infty}^t e^{-\delta a(t-s)} dW_s^H, \tag{1.9}$$

whose natural time scale is $1/\delta$ and whose variance is order one and given by $\sigma_{\rm ou}$ defined by (2.5) below, independently of δ. Then the volatility is

$$\sigma_t = F(Z_t^{\delta,H}), \tag{1.10}$$

where F is a smooth, positive-valued function, bounded away from zero, with bounded derivatives. We introduce the two parameters

$$\sigma_0 = F(Z_0^{\delta,H}), \qquad p_0 = F'(Z_0^{\delta,H}), \tag{1.11}$$

that is, the local level and rate of change of the volatility. In this case the implied volatility is given by:

$$I_t = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T \sigma_s^2 ds|\mathcal{F}_t\Big]^{\frac{1}{2}} + \frac{\delta^H p_0 \rho \tau_0^H}{\sqrt{2}\Gamma(H+\frac{5}{2})}\Big[(\tau/\tau_0)^{\frac{1}{2}+H} + (\tau/\tau_0)^{-\frac{1}{2}+H}\log(K/X_t)\Big], \tag{1.12}$$

for $\tau_0 = 2/\sigma_0^2$. Thus, the slow fractional volatility factor yields an implied volatility that corresponds to the one of the fractional model in (1.2) in the regime of small maturity, as given in (1.5). In the special case that $H = 1/2$ the volatility process becomes a standard Ornstein-Uhlenbeck process and is in the class of slow processes considered in [25] and indeed the implied volatility in (1.12) can then be show to be exactly of the form discussed for the slow correction in [25] (Chapter 5).

The outline of the paper is as follows. First in Section 2 we introduce the details of the ingredients of the fSV model. In Section 3 we derive the main result of the paper, the leading order expression for the price in the situation with a fSV. The derivation is based on a contract with a smooth payoff function while the European payoff function has a kink singularity and we generalize the result to this situation in Section 4. Then in Section 5 we derive the expression for the implied volatility and how the fractional character of the volatility affects this. We connect to the slow time volatility model in Section 6 and present some concluding remarks in Section 7. In Appendix A we characterize some quantities of interest and associated technical lemmas that are being used in the price derivation in Section 3.

2. The fractional stochastic volatility model. We describe in more detail the fBm and fOU processes that are used in the fSV construction (1.2).

2.1. Fractional Brownian motion and its moving-average stochastic integral representation. A fractional Brownian motion (fBm) is a zero-mean Gaussian process $(W_t^H)_{t\in\mathbb{R}}$ with the covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{\sigma_H^2}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \tag{2.1}$$

where σ_H is a positive constant.

We use the following moving-average stochastic integral representation of the fBm [36]:

$$W_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}} (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dW_s, \tag{2.2}$$

where $(W_t)_{t\in\mathbb{R}}$ is a standard Brownian motion over $\mathbb{R}$. In this model $(W_t^H)_{t\in\mathbb{R}}$ is a zero-mean Gaussian process with the covariance (2.1) where

$$\begin{aligned}\sigma_H^2 &= \frac{1}{\Gamma(H+\frac{1}{2})^2}\Big[\int_0^\infty \big((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\big)^2 ds + \frac{1}{2H}\Big]\\ &= \frac{1}{\Gamma(2H+1)\sin(\pi H)}. \end{aligned}\tag{2.3}$$

2.2. The fractional Ornstein-Uhlenbeck process. We then introduce the fractional Ornstein-Uhlenbeck process (fOU) as

$$Z_t^H = \int_{-\infty}^t e^{-a(t-s)} dW_s^H = W_t^H - a\int_{-\infty}^t e^{-a(t-s)} W_s^H ds. \tag{2.4}$$

It is a zero-mean, stationary Gaussian process, with variance

$$\sigma_{\rm ou}^2 = \mathbb{E}[(Z_t^H)^2] = \frac{1}{2}a^{-2H}\Gamma(2H+1)\sigma_H^2, \tag{2.5}$$

and covariance:

$$\begin{aligned}\mathbb{E}[Z_t^H Z_{t+s}^H] &= \sigma_{\rm ou}^2 \frac{1}{\Gamma(2H+1)}\Big[\frac{1}{2}\int_{\mathbb{R}} e^{-|v|}|as+v|^{2H}dv - |as|^{2H}\Big]\\ &= \sigma_{\rm ou}^2 \frac{2\sin(\pi H)}{\pi}\int_0^\infty \cos(asx)\frac{x^{1-2H}}{1+x^2}dx. \end{aligned}\tag{2.6}$$

Note that it is not a martingale, neither a Markov process.

The moving-average integral representation of the fOU is then

$$Z_t^H = \int_{-\infty}^t \mathcal{K}(t-s)dW_s, \tag{2.7}$$

where

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\Big[t^{H-\frac{1}{2}} - a\int_0^t (t-s)^{H-\frac{1}{2}}e^{-as}ds\Big]. \tag{2.8}$$

The properties of the kernel $\mathcal{K}$ are the following ones:
- $\mathcal{K}$ is nonnegative-valued, $\mathcal{K}\in L^2(0,\infty)$ for any $H\in(0,1)$ with $\int_0^\infty \mathcal{K}^2(u)du = \sigma_{\rm ou}^2$, and $\mathcal{K}\in L^1(0,\infty)$ for any $H\in(0,1/2)$.
- for small times $at \ll 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})a^{H-\frac{1}{2}}}\Big((at)^{H-\frac{1}{2}} + o\big((at)^{H-\frac{1}{2}}\big)\Big), \tag{2.9}$$

- for large times $at \gg 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H-\frac{1}{2})a^{H-\frac{1}{2}}}\Big((at)^{H-\frac{3}{2}} + o\big((at)^{H-\frac{3}{2}}\big)\Big), \tag{2.10}$$

For $H \in (0, 1/2)$ the fOU process possesses short-range correlation properties:

$$\mathbb{E}[Z_t^H Z_{t+s}^H] = \sigma_{\rm ou}^2 \Big(1 - \frac{1}{\Gamma(2H+1)} (as)^{2H} + o\big((as)^{2H}\big) \Big), \qquad as \ll 1. \tag{2.11}$$

For $H \in (1/2, 1)$ it possesses long-range correlation properties:

$$\mathbb{E}[Z_t^H Z_{t+s}^H] = \sigma_{\rm ou}^2 \Big(\frac{1}{\Gamma(2H-1)} (as)^{2H-2} + o\big((as)^{2H-2}\big) \Big), \qquad as \gg 1. \tag{2.12}$$

The expansion (2.12) is valid for any $H \in (0, 1/2) \cup (1/2, 1)$ and for $H \in (1/2, 1)$ it shows that the correlation function is not integrable at infinity. This is in contrast to the case of short-range dependent processes and also to Markov processes for which the correlation function is integrable.

3. The option price. The price of the risky asset follows the stochastic differential equation:

$$dX_t = \sigma_t X_t dW_t^*, \tag{3.1}$$

where the stochastic volatility is

$$\sigma_t = \bar{\sigma} + F(\delta Z_t^H), \tag{3.2}$$

Z_t^H has been introduced in the previous section and is adapted to the Brownian motion W_t, and W_t^* is a Brownian motion that is correlated to the stochastic volatility through

$$W_t^* = \rho W_t + \sqrt{1-\rho^2} B_t, \tag{3.3}$$

where the Brownian motion B_t is independent of W_t.

The function F is assumed to be one-to-one, smooth, bounded from below by a constant larger than $-\bar{\sigma}$, with bounded derivatives, and such that $F(0) = 0$ and $F'(0) = 1$. Accordingly, the filtration $\mathcal{F}_t$ generated by (B_t, W_t) is also the one generated by X_t. Indeed, it is equivalent to the one generated by (W_t^*, W_t), or (W_t^*, Z_t^H). Since F is one-to-one, it is equivalent to the one generated by (W_t^*, σ_t). Since $\bar{\sigma} + F$ is positive-valued, it is equivalent to the one generated by (W_t^*, σ_t^2), or X_t.

We aim at computing the option price defined as the martingale

$$M_t = \mathbb{E}\big[h(X_T) | \mathcal{F}_t \big], \tag{3.4}$$

where h is a smooth function. In fact weaker assumptions are possible for h, as we only need to control the function $Q_t^{(0)}(x)$ defined below rather than h, see Section 4.

The idea of the proof that we present below is to construct an approximation for M_t which has the correct terminal condition and which up to small (order δ^2) terms is a martingale. In then follows that we have a price approximation to $O(\delta^2)$.

We introduce the operator

$$\mathcal{L}_{\rm BS}(\sigma) = \partial_t + \frac{1}{2}\sigma^2 x^2 \partial_x^2. \tag{3.5}$$

The following proposition gives the first-order correction to the expression of the matingale M_t when δ is small.

Proposition 3.1. *When δ is small, we have*

$$M_t = Q_t(X_t) + O(\delta^2), \tag{3.6}$$

where

$$Q_t(x) = Q_t^{(0)}(x) + \delta\bar{\sigma}\phi_t\big(x^2\partial_x^2 Q_t^{(0)}(x)\big) + \delta\rho Q_t^{(1)}(x), \tag{3.7}$$

$Q_t^{(0)}(x)$ is deterministic and given by the Black-Scholes formula with constant volatility $\bar{\sigma}$,

$$\mathcal{L}_{\rm BS}(\bar{\sigma})Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x), \tag{3.8}$$

ϕ_t is the random component

$$\phi_t = \mathbb{E}\Big[\int_t^T Z_s^H ds | \mathcal{F}_t\Big], \tag{3.9}$$

and $Q_t^{(1)}(x)$ is the deterministic correction

$$Q_t^{(1)}(x) = \bar{\sigma}^2 x\partial_x\big(x^2\partial_x^2 Q_t^{(0)}(x)\big)D_{t,T}, \tag{3.10}$$

with $D_{t,T}$ defined by

$$D_{t,T} = \mathcal{D}(T-t), \qquad \mathcal{D}(\tau) = \frac{\tau^{H+\frac{3}{2}}}{\Gamma(H+\frac{5}{2})}\Big\{1 - \int_0^{a\tau} e^{-v}\big(1-\frac{v}{a\tau}\big)^{H+\frac{3}{2}}dv\Big\}. \tag{3.11}$$

Proof. For any smooth function $q_t(x)$, we have by Itô's formula

$$\begin{aligned} dq_t(X_t) &= \partial_t q_t(X_t)dt + \big(x\partial_x q_t\big)(X_t)\sigma_t dW_t^* + \frac{1}{2}\big(x^2\partial_x^2 q_t\big)(X_t)\sigma_t^2 dt \\ &= \mathcal{L}_{\rm BS}(\sigma_t)q_t(X_t)dt + \big(x\partial_x q_t\big)(X_t)\sigma_t dW_t^*, \end{aligned}$$

the last term being a martingale. Therefore, by (3.8), we have

$$dQ_t^{(0)}(X_t) = \big(\delta\bar{\sigma}Z_t^H + \frac{\delta^2}{2}g^\delta(Z_t^H)\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt + dN_t^{(0)}, \tag{3.12}$$

with $N_t^{(0)}$ a martingale,

$$dN_t^{(0)} = \big(x\partial_x Q_t^{(0)}\big)(X_t)\sigma_t dW_t^*,$$

and $g^\delta(y)$ is the function

$$g^\delta(y) = 2\bar{\sigma}\frac{F(\delta y) - \delta y}{\delta^2} + \frac{F(\delta y)^2}{\delta^2},$$

that can bounded uniformly in δ by

$$|g^\delta(y)| \le \big(\bar{\sigma}\|F''\|_\infty + \|F'\|_\infty^2\big)y^2.$$

Let ϕ_t be defined by (3.9). We have

$$\phi_t = \psi_t - \int_0^t Z_s^H ds, \tag{3.13}$$

where the martingale ψ_t is defined by

$$\psi_t = \mathbb{E}\Big[\int_0^T Z_s^H ds|\mathcal{F}_t\Big], \tag{3.14}$$

and it is studied in Appendix A. We can write

$$Z_t^H\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt = \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t - \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\phi_t.$$

By Itô's formula:

$$\begin{aligned}
d\big(\phi_t\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big) &= \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\phi_t \\
&\quad+\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t\phi_t dW_t^* \\
&\quad+\frac{1}{2}\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^2\phi_t dt \\
&\quad+\big(x^2\partial_x^2\partial_t\big)Q_t^{(0)}(X_t)\phi_t dt \\
&\quad+\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t d\left\langle \phi, W^*\right\rangle_t \\
&= \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\phi_t \\
&\quad+\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t\phi_t dW_t^* \\
&\quad+\big(\delta\bar{\sigma}Z_t^H + \frac{1}{2}\delta^2 g^\delta(Z_t^H)\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t dt \\
&\quad+\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t d\left\langle \phi, W^*\right\rangle_t,
\end{aligned}$$

where we have used again $\mathcal{L}_{\mathrm{BS}}(\bar{\sigma})Q_t^{(0)}(x) = 0$. We have $\left\langle \phi, W^*\right\rangle_t = \rho\left\langle \psi, W\right\rangle_t$ and therefore

$$\begin{aligned}
d\big(\phi_t\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big) &= -Z_t^H\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt \\
&\quad+\big(\delta\bar{\sigma}Z_t^H + \frac{1}{2}\delta^2 g^\delta(Z_t^H)\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t dt \\
&\quad+\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t d\left\langle \psi, W\right\rangle_t \\
&\quad+dN_t^{(1)},
\end{aligned}$$

where $N_t^{(1)}$ is a martingale,

$$dN_t^{(1)} = \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t\phi_t dW_t^* + \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t.$$

Therefore:

$$\begin{aligned}
&d\big(Q_t^{(0)}(X_t) + \delta\bar{\sigma}\phi_t\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big) \\
&= \big(\delta^2\bar{\sigma}^2 Z_t^H + \frac{1}{2}\delta^3\bar{\sigma}g^\delta(Z_t^H)\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t dt \\
&\quad+\frac{\delta^2}{2}g^\delta(Z_t^H)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt + \delta\bar{\sigma}\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t d\left\langle \psi, W\right\rangle_t \\
&\quad+dN_t^{(0)} + \bar{\sigma}\delta dN_t^{(1)}.
\end{aligned} \tag{3.15}$$

The deterministic function $Q_t^{(1)}$ defined by (3.10) satisfies

$$\mathcal{L}_{\mathrm{BS}}(\bar{\sigma})Q_t^{(1)}(x) = -\bar{\sigma}^2\big(x\partial_x\big(x^2\partial_x^2 Q_t^{(0)}(x)\big)\big)\theta_{t,T}, \qquad Q_T^{(1)}(x) = 0,$$

where $\theta_{t,T}$ is such that

$$d\langle\psi, W\rangle_t = \theta_{t,T}dt,$$

and it is given by (see Lemma A.1):

$$\theta_{t,T} = \int_t^T \mathcal{K}(v-t)dv = \int_0^{T-t} \mathcal{K}(v)dv.$$

Applying Itô's formula

$$\begin{aligned}
dQ_t^{(1)}(X_t) &= \mathcal{L}_{\mathrm{BS}}(\sigma_t)Q_t^{(1)}(X_t)dt + \big(x\partial_x Q_t^{(1)}\big)(X_t)\sigma_t dW_t^* \\
&= \mathcal{L}_{\mathrm{BS}}(\bar{\sigma})Q_t^{(1)}(X_t)dt + \big(\delta\bar{\sigma}Z_t^H + \frac{\delta^2}{2}g^\delta(Z_t^H)\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt \\
&\quad + \big(x\partial_x Q_t^{(1)}\big)(X_t)\sigma_t dW_t^* \\
&= -\bar{\sigma}^2\big(x\partial_x\big(x^2\partial_x^2 Q_t^{(0)}(x)\big)\big)d\langle\psi, W\rangle_t \\
&\quad + \big(\delta\bar{\sigma}Z_t^H + \frac{\delta^2}{2}g^\delta(Z_t^H)\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt + dN_t^{(2)},
\end{aligned}$$

where $N_t^{(2)}$ is a martingale,

$$dN_t^{(2)} = \big(x\partial_x Q_t^{(1)}\big)(X_t)\sigma_t dW_t^*.$$

Therefore

$$d\big(Q_t^{(0)}(X_t) + \delta\bar{\sigma}\phi_t\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t) + \delta\rho Q_t^{(1)}(X_t)\big) = dR_t + dN_t, \tag{3.16}$$

where N_t is a martingale,

$$dN_t = dN_t^{(0)} + \bar{\sigma}\delta dN_t^{(1)} + \rho\delta dN_t^{(2)}, \tag{3.17}$$

and R_t is of order δ^2:

$$\begin{aligned}
dR_t = &\big(\delta^2\bar{\sigma}^2 Z_t^H + \frac{1}{2}\delta^3\bar{\sigma}g^\delta(Z_t^H)\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t dt \\
&+\frac{\delta^2}{2}g^\delta(Z_t^H)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt + \delta^2\bar{\sigma}\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)Z_t^H\theta_{t,T}dt \\
&+\big(\delta^2\rho\bar{\sigma}Z_t^H + \frac{\delta^3}{2}\rho g^\delta(Z_t^H)\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt.
\end{aligned} \tag{3.18}$$

Then with $Q_t(x)$ defined as in Proposition 3.1 we have $Q_T(x) = h(x)$ because $Q_T^{(0)}(x) = h(x)$, $\phi_T = 0$, and $Q_T^{(1)}(x) = 0$. Therefore

$$\begin{aligned}
M_t &= \mathbb{E}\big[h(X_T)|\mathcal{F}_t\big] = \mathbb{E}\big[Q_T(X_T)|\mathcal{F}_t\big] = \mathbb{E}\big[N_T|\mathcal{F}_t\big] + \mathbb{E}\big[R_T|\mathcal{F}_t\big] \\
&= N_t^{(4)} + \mathbb{E}\big[R_T|\mathcal{F}_t\big] = Q_t(X_t) + \mathbb{E}\big[R_T - R_t|\mathcal{F}_t\big],
\end{aligned} \tag{3.19}$$

which completes the proof since $\mathbb{E}\big[R_T - R_t|\mathcal{F}_t\big]$ is of order δ^2. □

4. Accuracy with European option. In the derivation above we assumed a smooth payoff function. Since important classes of payoff functions have non-smooth payoff we generalize here the proof to such a class by considering a European option. For a European option $h(x) = (x-K)_+$ we have [25]

$$Q_t^{(0)}(x) = x\Phi\Big(\frac{1}{\bar{\sigma}\sqrt{T-t}}\log\big(\frac{x}{K}\big) + \frac{\bar{\sigma}\sqrt{T-t}}{2}\Big)$$
$$-K\Phi\Big(\frac{1}{\bar{\sigma}\sqrt{T-t}}\log\big(\frac{x}{K}\big) - \frac{\bar{\sigma}\sqrt{T-t}}{2}\Big), \tag{4.1}$$

where Φ is the cumulative distribution function of the standard normal distribution. We can see that h is not smooth so that the hypotheses of Proposition 3.1 are not satisfied. However the conclusions of Proposition 3.1 still hold true as we now show.

Proof. One has to show that R_t defined by (3.18) satisfies $\mathbb{E}\big[R_T - R_t|\mathcal{F}_t\big]$ is of order δ^2 in L^p for any p and that the local martingale N_t defined by (3.17) is a martingale (up to time T). The problem comes from the fact that the derivatives of $Q_t^{(0)}(x)$ blow up when $t \to T$. However this blow up is not strong as we show below. We first state a few properties of the deterministic and random terms that appear in the expression of R_t:

- The deterministic function $Q_t^{(0)}(x)$ given by (4.1) satisfies

$$\partial_x^k Q_t^{(0)}(x) \le C\Big(1 + \frac{1}{(T-t)^{\frac{k-1}{2}}}\Big),$$

for any $1 \le k \le 4$, $t \in [0,T]$, $x \in (0,\infty)$, and for some constant C.

- The deterministic quantity $D_{t,T}$ satisfies

$$D_{t,T} \le C(T-t)^{H+\frac{3}{2}},$$

for any $t \in [0,T]$ and for some constant C.

- The deterministic quantity $\theta_{t,T}$ satisfies

$$\theta_{t,T} \le C(T-t)^{H+\frac{1}{2}},$$

for any $t \in [0,T]$ and for some constant C.

- The random component ϕ_t satisfies

$$\mathbb{E}[|\phi_t|^p]^{\frac{1}{p}} \le C_p(T-t),$$

for any $t \in [0,T]$ and for some constant C_p for any $p > 0$.

- The random process Z_t^H satisfies

$$\mathbb{E}[|Z_t^H|^p]^{\frac{1}{p}} \le C_p,$$

for any $t \in [0,T]$ and for some constant C_p for any $p > 0$.

As a consequence, the deterministic function $Q_t^{(1)}(x)$ satisfies

$$|\partial_x^k Q_t^{(1)}(x)| \le C\Big((T-t)^{H+\frac{3}{2}} + (T-t)^{H+\frac{1}{2}-\frac{k}{2}}\Big)\big(1+x^3\big),$$

for any $1 \leq k \leq 2$, $t \in [0,T]$, $x \in (0,\infty)$, and for some constant C.
Using (3.18) we have

$$
\begin{aligned}
R_T - R_t = \delta^2 \int_t^T & \big(\bar\sigma^2 Z_s^H + \frac{1}{2}\delta\bar\sigma g^\delta(Z_s^H)\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_s^{(0)}(X_s)\phi_s ds \\
&+\delta^2 \int_t^T \frac{1}{2} g^\delta(Z_s^H)\big(x^2\partial_x^2\big)Q_s^{(0)}(X_s)ds \\
&+\delta^2 \int_t^T \bar\sigma\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_s^{(0)}(X_s)Z_s^H\theta_{s,T}ds \\
&+\delta^2 \int_t^T \big(\rho\bar\sigma Z_s^H + \frac{\delta}{2}\rho g^\delta(Z_s^H)\big)\big(x^2\partial_x^2\big)Q_s^{(1)}(X_s)ds.
\end{aligned}
$$

Using the previous estimates we find that, for any $p>0$, there exists a constant C_p such that

$$
\begin{aligned}
\mathbb{E}[|R_T - R_t|^p]^{\frac{1}{p}} &\leq C_p\delta^2 \int_t^T (T-s)^{-\frac{1}{2}} + (T-s)^{-\frac{1}{2}} + (T-s)^{H-\frac{1}{2}} + (T-s)^{H-\frac{1}{2}}ds \\
&\leq C_p\delta^2\big((T-t)^{\frac{1}{2}} + (T-t)^{H+\frac{1}{2}}\big),
\end{aligned}
$$

for any $\delta \in (0,1)$ and $t \in [0,T]$, which shows the desired result for R.
Moreover, the local martingales $N_t^{(j)}$ in (3.17) are continuous square-integrable martingales up to time T whose brackets are

$$
\begin{aligned}
d\left\langle N^{(j)}\right\rangle_t &= \mathcal{N}_t^{(j)}dt, \quad j=0,1,2, \\
\mathcal{N}_t^{(0)} &= \big(\sigma_t\big(x\partial_x Q_t^{(0)}\big)(X_t)\big)^2, \\
\mathcal{N}_t^{(1)} &= \big(\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t\phi_t\big)^2 \\
&\quad +2\rho\theta_{t,T}\big(\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t\phi_t\big)\big(\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big) \\
&\quad +\big(\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big)^2\theta_{t,T}^2, \\
\mathcal{N}_t^{(2)} &= \big(\sigma_t\big(x\partial_x Q_t^{(1)}\big)(X_t)\big)^2,
\end{aligned}
$$

where the $\mathcal{N}_t^{(j)}$ are uniformly bounded with respect to $t \in [0,T]$ in L^p for any p which concludes the proof. □

5. The implied volatility. The implied volatility in the context of the European option introduced in the previous section is given by

$$
I_t = \bar\sigma + \delta\frac{\phi_t}{T-t} + \delta\rho D_{t,T}\Big[\frac{\bar\sigma}{2(T-t)} + \frac{\log(K/X_t)}{\bar\sigma(T-t)^2}\Big] + O(\delta^2). \tag{5.1}
$$

The first two terms can be combined and rewritten as (up to terms of order δ^2):

$$
\bar\sigma + \delta\frac{\phi_t}{T-t} = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T \sigma_s^2 ds|\mathcal{F}_t\Big]^{\frac{1}{2}} + O(\delta^2). \tag{5.2}
$$

When $a(T-t) \ll 1$ the implied volatility is random and we have (see Lemma A.3) and Eq. (A.5) :

$$
I_t = \bar\sigma + \delta Z_t^H + \delta\frac{\rho}{\Gamma(H+\frac{5}{2})}\Big[\frac{\bar\sigma}{2}(T-t)^{\frac{1}{2}+H} + \frac{\log(K/X_t)}{\bar\sigma(T-t)^{\frac{1}{2}-H}}\Big]. \tag{5.3}
$$

Note that, for $H \in (0,1/2)$, the implied volatility blows up at small time-to-maturity $T-t$.

When $a(T-t) \gg 1$, the quantity $D_{t,T}$ is of order $(T-t)^{H+\frac{1}{2}}$ and is deterministic (by Lemma A.2), while the fluctuations of ϕ_t are of order $(T-t)^H$ at most and are therefore negligible (by Lemma A.3). As a consequence, when $a(T-t) \gg 1$, we can write the implied volatility as:

$$I_t = \bar{\sigma} + \delta \frac{\rho}{a\Gamma(H+\frac{3}{2})} \Big[\frac{\bar{\sigma}}{2}(T-t)^{H-\frac{1}{2}} + \frac{\log(K/X_t)}{\bar{\sigma}(T-t)^{\frac{3}{2}-H}} \Big]. \tag{5.4}$$

Note that, for $H \in (1/2,1)$, the implied volatility blows up at large time-to-maturity $T-t$.

6. A slow volatility factor. We show in this section that the approach developed in this paper can be applied to other stochastic volatility models. Here we consider the following model

$$\sigma_t = F(Z_t^{\delta,H}), \tag{6.1}$$

where F is a smooth, positive-valued function, bounded away from zero, with bounded derivatives, and $Z_t^{\delta,H}$ is a rescaled fOU process:

$$dZ_t^{\delta,H} = \delta^H dW_t^H - \delta a Z_t^{\delta,H} dt, \tag{6.2}$$

whose natural time scale is $1/\delta$. It has the form

$$Z_t^{\delta,H} = \delta^H \int_{-\infty}^t e^{-\delta a(t-s)} dW_s^H. \tag{6.3}$$

Its moving-average integral representation is

$$Z_t^{\delta,H} = \int_{-\infty}^t \mathcal{K}^\delta(t-s) dW_s, \qquad \mathcal{K}^\delta(t) = \delta^{\frac{1}{2}} \mathcal{K}(\delta t), \tag{6.4}$$

where $\mathcal{K}$ is defined by (2.8). In particular its variance is $\sigma_{\rm ou}$ defined by (2.5), independently of δ. This model is therefore characterized by strong but slow fluctuations of the volatility. If the price of the risky asset follows the stochastic differential equation (3.1), we get a result similar to Proposition 3.1.

PROPOSITION 6.1. *When δ is small, denoting $\sigma_0 = F(Z_0^{\delta,H})$ and $p_0 = F'(Z_0^{\delta,H})$, the option price (3.4) is of the form*

$$M_t = Q_t(X_t) + O(\delta^{2H}), \tag{6.5}$$

where

$$Q_t(x) = Q_t^{(0)}(x) + \sigma_0 p_0 \phi_t^\delta \big(x^2 \partial_x^2 Q_t^{(0)}(x)\big) + \delta^H \rho p_0 Q_t^{(1)}(x), \tag{6.6}$$

$Q_t^{(0)}(x)$ is given by the Black-Scholes formula with constant volatility σ_0,

$$\mathcal{L}_{\rm BS}(\sigma_0) Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x), \tag{6.7}$$

ϕ_t^δ is the random component

$$\phi_t^\delta = \mathbb{E}\Big[\int_t^T Z_s^{\delta,H} - Z_0^{\delta,H} ds | \mathcal{F}_t \Big], \tag{6.8}$$

and $Q_t^{(1)}(x)$ *is the correction*

$$Q_t^{(1)}(x) = \sigma_0^2 x\partial_x\big(x^2\partial_x^2 Q_t^{(0)}(x)\big)D_{t,T}, \tag{6.9}$$

with $D_{t,T}$ *defined by*

$$D_{t,T} = \frac{(T-t)^{H+\frac{3}{2}}}{\Gamma(H+\frac{5}{2})}. \tag{6.10}$$

The random correction ϕ_t^δ is of order δ^H. More exactly it is a zero-mean Gaussian random variable with variance

$$\mathbb{E}\big[(\phi_t^\delta)^2\big] = \frac{\delta^{2H}T^{2+2H}}{\Gamma(H+\frac{3}{2})^2}\int_0^\infty \big[(1-\alpha+v)^{H+\frac{1}{2}} - v^{H+\frac{1}{2}} - (1-\alpha)(H+\frac{1}{2})(v-\alpha)_+^{H-\frac{1}{2}}\big]^2 dv, \tag{6.11}$$

for $t = \alpha T$ and $\alpha \in [0,1]$.

Proof. We note that

$$\sigma_t = \sigma_0 + p_0(Z_t^{\delta,H} - Z_0^{\delta,H}) + g_t^\delta,$$

where $g_t^\delta = F(Z_t^{\delta,H}) - F(Z_0^{\delta,H}) - F'(Z_0^{\delta,H})(Z_t^{\delta,H} - Z_0^{\delta,H})$ and therefore

$$|g_t^\delta| \le \frac{1}{2}\|F''\|_\infty (Z_t^{\delta,H} - Z_0^{\delta,H})^2.$$

We have

$$\mathbb{E}\big[(Z_t^{\delta,H} - Z_0^{\delta,H})^2\big] = \int_0^{\delta t}\mathcal{K}(s)^2 ds + \int_0^\infty \big[\mathcal{K}(\delta t + s) - \mathcal{K}(s)\big]^2 ds,$$

which is of order δ^{2H}:

$$\mathbb{E}\big[(Z_t^{\delta,H} - Z_0^{\delta,H})^2\big] = \sigma_H^2(\delta t)^{2H} + o(\delta^{2H}).$$

Therefore g_t^δ is bounded in L^p for any p by a quantity of order δ^{2H}. We can then follow the same proof as the one of Proposition 3.1. The term

$$D_{t,T}^\delta = \int_0^\tau (\tau - u)\mathcal{K}^\delta(u)du,$$

is given by

$$D_{t,T}^\delta = \delta^H\frac{(T-t)^{H+\frac{3}{2}}}{\Gamma(H+\frac{5}{2})} + O(\delta^{2H}).$$

The variance of the correction ϕ_t^δ is

$$\mathbb{E}\big[(\phi_t^\delta)^2\big] = \int_0^t \Big(\int_t^T \mathcal{K}^\delta(s-u)ds\Big)^2 du + \int_{-\infty}^0 \Big(\int_t^T \mathcal{K}^\delta(s-u) - \mathcal{K}^\delta(-u)ds\Big)^2 du,$$

which in turn gives (6.11). □

Proceeding as in the case of the small-amplitude stochastic volatiliy model, we find that the implied volatility in the context of the European option is given by

$$I_t = \sigma_0 + p_0\frac{\phi_t^\delta}{T-t} + \delta^H\frac{\rho p_0}{\Gamma(H+\frac{5}{2})}\Big[\frac{\sigma_0}{2}(T-t)^{H+\frac{1}{2}} + \frac{\log(K/X_t)}{\sigma_0(T-t)^{\frac{1}{2}-H}}\Big] + O(\delta^{2H}). \tag{6.12}$$

The first two terms can be combined and rewritten as (up to terms of order δ^{2H}):

$$\sigma_0 + p_0\frac{\phi_t^\delta}{T-t} = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T \sigma_s^2 ds|\mathcal{F}_t\Big]^{\frac{1}{2}} + O(\delta^{2H}). \tag{6.13}$$

7. Conclusion. We have presented an analysis of the European option price when the volatility is stochastic and has correlations that decay as a fractional power of the time offset. The stochastic volatility model is defined in terms of a fractional Ornstein Uhlenbeck process with Hurst exponent H and the analysis is carried out when the typical amplitude of the volatility fluctuations is relatively small. Two situations are differentiated. First the situation when $H \in (0, 1/2)$ which corresponds to a "short-range" dependent process that is rough on short scales with correlations that decay very rapidly, faster than linear decay, at the origin. Second the situation when $H \in (1/2, 1)$ so that the correlations decay relatively slowly at large scales and then the volatility correlations are not integrable. We use a martingale method approach to derive a general expression for the Black-Scholes price covering the two cases. In the short-range case the rough behavior on short scales gives rise to an implied volatility that diverges as the time to maturity goes to zero. In the long-range case the slow decay in the correlations gives a term structure of the implied volatility that diverges as time to maturity goes to infinity. The main result we have presented is specific in the sense that a particular stochastic volatility model has been addressed, however, as we illustrate the framework can be adapted to related models as long as some central covariance terms can be computed. We illustrate this by considering a model with slow, but order one, volatility fluctuations and derive the associated fractional implied volatility term structure.

Appendix A. Technical lemmas. In this appendix we state and prove a few technical lemmas related to some central quantities of interest that are used in the derivation of the price in Sections 3 and 5.

The martingale ψ_t is defined for any $t \in [0, T]$ by (3.14). It is used in the proof of Proposition 3.1 and it has the following properties.

LEMMA A.1. $(\psi_t)_{t\in[0,T]}$ *is a Gaussian square-integrable martingale and*

$$d\left\langle \psi, W \right\rangle_t = \Big(\int_0^{T-t} \mathcal{K}(s) ds \Big) dt, \qquad d\left\langle \psi \right\rangle_t = \Big(\int_0^{T-t} \mathcal{K}(s) ds \Big)^2 dt. \tag{A.1}$$

Proof. For $t \le s$, the conditional distribution of Z_s^H given $\mathcal{F}_t$ is Gaussian with mean

$$\mathbb{E}\big[Z_s^H | \mathcal{F}_t \big] = \int_{-\infty}^t \mathcal{K}(s-u) dW_u, \tag{A.2}$$

and deterministic variance given by

$$\mathrm{Var}\big(Z_s^H | \mathcal{F}_t \big) = \int_0^{s-t} \mathcal{K}(u)^2 du.$$

Therefore we have

$$\begin{aligned}
\psi_t &= \int_0^t Z_s^H ds + \int_t^T \mathbb{E}\big[Z_s^H | \mathcal{F}_t \big] ds \\
&= \int_0^t ds \int_{-\infty}^s \mathcal{K}(s-u) dW_u + \int_t^T dt \int_{-\infty}^t \mathcal{K}(s-u) dW_u \\
&= \int_{-\infty}^0 \Big[\int_0^T \mathcal{K}(s-u) dt \Big] dW_u + \int_0^t \Big[\int_u^T \mathcal{K}(s-u) dt \Big] dW_u .
\end{aligned}$$

This gives

$$d\left\langle \psi, W\right\rangle_t = \Big(\int_t^T \mathcal{K}(s-t)ds\Big)dt, \qquad d\left\langle \psi\right\rangle_t = \Big(\int_t^T \mathcal{K}(s-t)ds\Big)^2 dt,$$

as stated in the Lemma. □

We define the deterministic component

$$D_{t,T} = \left\langle \psi, W\right\rangle_T - \left\langle \psi, W\right\rangle_t, \tag{A.3}$$

that appears in Equation (3.10). It has the following properties.

LEMMA A.2. *$D_{t,T}$ is a deterministic function of $T-t$ and it is given by*

$$D_{t,T} = \mathcal{D}(T-t), \qquad \mathcal{D}(\tau) = \int_0^\tau (\tau-u)\mathcal{K}(u)du. \tag{A.4}$$

The function $\mathcal{D}$ can be written as (3.11) and it has the following behavior:
For $a\tau \ll 1$,

$$\mathcal{D}(\tau) = \frac{1}{\Gamma(H+\frac{5}{2})a^{H+\frac{3}{2}}}\Big((a\tau)^{H+\frac{3}{2}} + o\big((a\tau)^{H+\frac{3}{2}}\big)\Big). \tag{A.5}$$

For $a\tau \gg 1$,

$$\mathcal{D}(\tau) = \frac{1}{\Gamma(H+\frac{3}{2})a^{H+\frac{3}{2}}}\Big((a\tau)^{H+\frac{1}{2}} + o\big((a\tau)^{H+\frac{1}{2}}\big)\Big). \tag{A.6}$$

Finally, we consider the random process ϕ_t defined by (3.9).

LEMMA A.3.

1. *ϕ_t is a zero-mean Gaussian process with variance*

$$\mathrm{Var}(\phi_t) = \int_0^\infty \Big(\int_0^{T-t} \mathcal{K}(s+u)ds\Big)^2 du.$$

2. *There exists a constant C (that depends on H) such that the variance of ϕ_t can be bounded by*

$$\mathrm{Var}(\phi_t) \le C\,(T-t)^{2H} \wedge (T-t)^2. \tag{A.7}$$

3. *ϕ_t is approximately equal to $(T-t)Z_t^H$ for small $T-t$:*

$$\mathbb{E}\Big[\Big(\frac{\phi_t}{T-t} - Z_t^H\Big)^2\Big] \stackrel{T-t\to 0}{\longrightarrow} 0. \tag{A.8}$$

Proof. We can express the variance of ϕ_t as:

$$\mathrm{Var}(\phi_t) = \int_0^{T-t} ds \int_0^{T-t} ds' \mathrm{Cov}\big(\mathbb{E}\big[Z_s^H|\mathcal{F}_0\big], \mathbb{E}\big[Z_{s'}^H|\mathcal{F}_0\big]\big),$$

which gives the first item since

$$\mathrm{Cov}\big(\mathbb{E}\big[Z_s^H|\mathcal{F}_0\big], \mathbb{E}\big[Z_{s'}^H|\mathcal{F}_0\big]\big) = \int_{-\infty}^0 \mathcal{K}(s-u)\mathcal{K}(s'-u)du.$$

Furthermore

$$\begin{aligned}\mathrm{Var}(\phi_t) &\leq \Big(\int_0^{T-t} \mathrm{Var}\big(\mathbb{E}\big[Z_s^H|\mathcal{F}_0\big]\big)^{1/2} ds\Big)^2 \\ &\leq \Big(\int_0^{T-t} \Big(\int_s^\infty \mathcal{K}(u)^2 du\Big)^{1/2} ds\Big)^2 \\ &\leq C\,(T-t)^{2H} \wedge (T-t)^2,\end{aligned}$$

which gives the second item of the lemma.

Similarly, we have

$$\mathbb{E}\Big[\Big(\frac{\phi_t}{T-t} - Z_t^H\Big)^2\Big] \leq \Big(\frac{1}{T-t}\int_0^{T-t} ds \mathrm{Var}\big(\mathbb{E}\big[Z_s^H|\mathcal{F}_0\big] - Z_0^H\big)^{1/2}\Big)^2,$$

and

$$\mathbb{E}\big[Z_s^H|\mathcal{F}_0\big] - Z_0^H = \int_{-\infty}^0 \big(\mathcal{K}(s-u) - \mathcal{K}(-u)\big) dW_u,$$

so that

$$\mathbb{E}\Big[\Big(\frac{\phi_t}{T-t} - Z_t^H\Big)^2\Big] \leq \Big(\frac{1}{T-t}\int_0^{T-t} \Big[\int_0^\infty \big(\mathcal{K}(s+v) - \mathcal{K}(v)\big)^2 dv\Big] ds\Big)^2.$$

As $s \to 0$, we have $\int_0^\infty \big(\mathcal{K}(s+v)-\mathcal{K}(v)\big)^2 dv \to 0$ by Lebesgue's dominated convergence theorem (remember $\mathcal{K} \in L^2$), which gives the third item. □

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